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# $(p, 1)$-Total labelling of graphs

Frédéric Havet[a], Min-Li Yu[b]

[a] Projet Mascotte, CNRS/UNSA/INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex, France

[b] Department of Mathematics and Statistics, University College of the Fraser Valley, Abbotsford, BC, Canada V2S 7M8

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**Abstract**

A $(p, 1)$-total labelling of a graph $G$ is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of $G$ receive distinct integers, (ii) any two adjacent edges of $G$ receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least $p$ in absolute value. The *span* of a $(p, 1)$-total labelling is the maximum difference between two labels. The minimum span of a $(p, 1)$-total labelling of $G$ is called the $(p, 1)$-*total number* and denoted by $\lambda_p^T(G)$.

We provide lower and upper bounds for the $(p, 1)$-total number. In particular, generalizing the Total Colouring Conjecture, we conjecture that $\lambda_p^T \leqslant \Delta + 2p - 1$ and give some evidences to support it. Finally, we determine the exact value of $\lambda_p^T(K_n)$, except for even $n$ in the interval $[p+5, 6p^2 - 10p + 4]$ for which we show that $\lambda_p^T(K_n) \in \{n + 2p - 3, n + 2p - 2\}$.



*Keywords:* $(p, 1)$-Total labelling; Minimum span; Maximum degree; Bounds for graphs

## 1. Introduction

In the channel assignment problem, the following situation occurs: we need to assign radio frequency bands to transmitters (each station gets one channel which corresponds to an integer). In order to avoid interference, if two stations are too close, then the separation of the channels assigned to them has to be at least two. Moreover, if two stations are close (but not too close), then they must receive different channels. Motivated by this problem, Griggs and Yeh [4] introduced $L(2, 1)$-labellings. Its natural generalization $L(p, 1)$-labellings of a graph $G$ is an integer assignment $L$ to the vertex set $V(G)$ such that:

$$|L(u) - L(v)| \geqslant p \text{ if } d_G(u, v) = 1 \quad \text{and} \quad |L(u) - L(v)| \geqslant 1 \text{ if } d_G(u, v) = 2.$$

This labelling has been studied in several articles. In [2] it was studied for chordal graphs. In particular, Whittlesey et al. [10] studied $L(2, 1)$-labellings of first subdivision of a graph $G$. The *first subdivision* of a graph $G$ is the graph $s_1(G)$ obtained from $G$ by inserting one vertex along each edge of $G$. An $L(p, 1)$-labelling of $s_1(G)$ corresponds to an assignment of integers to $V(G) \cup E(G)$ such that:

(i) any two adjacent vertices of $G$ receive distinct integers,

*E-mail addresses:* fhavet@sophia.inria.fr (F. Havet), joseph.yu@ucfv.ca (M.-L. Yu).

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(ii) any two adjacent edges of $G$ receive distinct integers, and
(iii) a vertex and an edge incident receive integers that differ by at least $p$ in absolute value.

We call such an assignment a $(p, 1)$-*total labelling of* $G$. It is a total colouring strengthened with an extra condition by insisting on a minimal separation of $p$ between incident vertices and edges.

The *span* of a $(p, 1)$-total labelling is the maximum difference between two labels. The $(p, 1)$-*total number* of a graph $G$, denoted by $\lambda_p^{\mathrm{T}}(G)$, is the minimum span of a $(p, 1)$-total labelling of $G$. Note that a $(1, 1)$-total labelling is a total colouring as $\lambda_1^{\mathrm{T}} = \chi^{\mathrm{T}} - 1$, where $\chi^{\mathrm{T}}$ is the total chromatic number. By generalizing the Total Colouring Conjecture, we conjecture that $\lambda_p^{\mathrm{T}} \leqslant \Delta + 2p - 1$ and call it the $(p, 1)$-Total Labelling Conjecture.

The aim of this paper is to study $(p, 1)$-total labellings of graphs and in particular, bounds for the $(p, 1)$-total number $\lambda_p^{\mathrm{T}}$ as a function of the maximum degree $\Delta$ of the graph.

In Section 2, we give some general bounds and show that $\lambda_p^{\mathrm{T}} \leqslant 2\Delta + p - 1$. Some evidences are provided to support the $(p, 1)$-Total Labelling Conjecture. By generalizing a result of [6], we show that if $\Delta(G)$ is large enough then $\lambda_p^{\mathrm{T}}(G) \leqslant \Delta(G) + \mathrm{O}(\log^{10} \Delta(G))$ and extending a result of [7], also show that as $n \to \infty$, the proportion of graphs on $n$ vertices with $(p, 1)$-total number $\lambda_p^{\mathrm{T}} > \Delta + 2p - 1$ is very small.

In Section 3, we show that $\lambda_p^{\mathrm{T}} \leqslant 2\Delta - 2\log(\Delta + 2) + 2\log(16p - 8) + p - 1$ which gives a better upper bound on the $(p, 1)$-total number when $\Delta$ is not too large.

In Section 4, we focus on the (2, 1)-total labelling. We show that if $\Delta \geqslant 2$, then $\lambda_2^{\mathrm{T}} \leqslant 2\Delta$ and therefore the $(p, 1)$-Total Labelling Conjecture is true when $p = 2$ and $\Delta = 3$. In fact, the bound for this special case is tight as $\lambda_2^{\mathrm{T}}(K_4) = 6$. We then improve this bound to $2\Delta - 1$ when $\Delta$ is odd and at least 5.

In Section 5, we discuss the tightness of some bounds.

Finally, in Section 6, we study the $(p, 1)$-total number of complete graphs and determine the exact values of the $(p, 1)$-total numbers for almost all complete graphs.

## 2. Some general bounds and the $(p, 1)$-Total Labelling Conjecture

Looking at the label of a vertex with maximum degree and its incident edges, it is easy to see that $\lambda_p^{\mathrm{T}} \geqslant \Delta + p - 1$. This lower bound may be increased in some cases.

**Proposition 1.** (i) *If $G$ is $\Delta$-regular then* $\lambda_p^{\mathrm{T}} \geqslant \Delta + p$.
(ii) *If* $p \geqslant \Delta$, *then* $\lambda_p^{\mathrm{T}} \geqslant \Delta + p$.

**Proof.** Suppose for a contradiction that $G$ admits a $(p, 1)$-labelling in $[0, \Delta + p - 1]$. Then every vertex must be labelled either 0 or $\Delta + p - 1$. Let $v$ be a vertex of $G$. Without loss of generality, we may suppose that $v$ is labelled 0. Then its incident edges are labelled with $\{p, p + 1, \dots, \Delta + p - 1\}$.

(i) Let $vw$ be the edge that is labelled $\Delta + p - 1$. Then $w$ cannot be labelled $\Delta + p - 1$ nor 0. This is a contradiction.
(ii) Let $vw$ be the edge that is labelled $p$. The vertex $w$ must have a label that is bigger than $2p - 1$, thus bigger than $\Delta + p - 1$. This is a contradiction. □

**Definition 2.** Let $G$ be a graph. The chromatic number and the chromatic index of $G$ are denoted by $\chi(G)$ and $\chi'(G)$, respectively. When $G$ is clear from the context, we simply write $\chi$ and $\chi'$ instead of $\chi(G)$ and $\chi'(G)$.

**Proposition 3.**

$$\lambda_p^{\mathrm{T}} \leqslant \chi + \chi' + p - 2,$$

$$\lambda_p^{\mathrm{T}} \leqslant 2\Delta + p - 1.$$

**Proof.** It suffices to prove the results for connected graphs. Assume that $G$ is a connected graph. Let $c$ be a vertex colouring of $G$ with the $\chi$ integers of $[0, \chi - 1]$, and $c'$ be an edge colouring of $G$ with the $\chi'$ integers of $[\chi - 1 + p, \chi + \chi' + p - 2]$. Then the union of $c$ and $c'$ is obviously a $(p, 1)$-labelling of $G$. Thus $\lambda_p^{\mathrm{T}} \leqslant \chi + \chi' + p - 2$.

If $G$ is neither a complete graph nor an odd cycle, then $\chi \leqslant \Delta$ by Brook's theorem and $\chi' \leqslant \Delta+1$ by Vizing's theorem. Hence, $\lambda_p^{\mathrm{T}} \leqslant 2\Delta + p - 1$.

Suppose now that $G$ is the complete graph $K_n$ on $n$ vertices. $\chi = n = \Delta+1$. If $n$ is even then $\chi' = \Delta$. So $\lambda_p^{\mathrm{T}} \leqslant 2\Delta + p - 1$. If $n$ is odd, then $\chi' = \Delta+1$. Let $c'$ be an edge colouring of $G$ with $n$ colours and $M_i$, $1 \leqslant i \leqslant n$, be the matchings corresponding to the colour classes, and furthermore, each $M_i$ contains all vertices but one $v_i$. For $1 \leqslant i \leqslant n$, label the vertex $v_i$ with $n-i$ and the edges of $M_i$ with $n+p-3+i$. Since $v_1$ is not incident to any edge of $M_1$, then we have a $(p,1)$-total labelling of $K_n$ in $[0, 2n+p-3] = [0, 2\Delta + p - 1]$.

If $G$ is an odd cycle, then label the vertices with 0, 1 and 2 such that exactly one vertex $v$ is assigned 2, and label the edges with 3, 4 and 5 such that exactly one edge $e$, not incident to $v$, is assigned 3. □

**Corollary 4.** *Let $G$ be a bipartite graph. Then $\Delta + p - 1 \leqslant \lambda_p^{\mathrm{T}} \leqslant \Delta + p$.*

*Moreover, if $p \geqslant \Delta$ or $G$ is regular then $\lambda_p^{\mathrm{T}} = \Delta + p$.*

**Proof.** If $G$ is bipartite, then $\chi = 2$ and $\chi' = \Delta$ by König's theorem. Then Propositions 1 and 3 give the result. □

**Remark 5.** If $p < \Delta$, there are bipartite graphs for which $\lambda_p^{\mathrm{T}} = \Delta + p - 1$ or $\lambda_p^{\mathrm{T}} = \Delta + p$. Havet and Thomassé [5] proved that it is NP-complete to decide the exact value of $\lambda_p^{\mathrm{T}}$ for a bipartite graph $G$.

As a natural extension of the Total Colouring Conjecture to $(p,1)$-total labelling, we conjecture the following.

$(p,1)$-**Total Labelling Conjecture.** $\lambda_p^{\mathrm{T}} \leqslant \Delta + 2p - 1$ or $\lambda_p^{\mathrm{T}} \leqslant \min\{\Delta + 2p - 1, 2\Delta + p - 1\}$.

Lots of upper bounds on the total chromatic number have been given and most of the proofs may be slightly modified to obtain upper bounds for $\lambda_p^{\mathrm{T}}$.

In [7], McDiarmid and Reed proved that given a graph $G$ with $n$ vertices, $\chi^{\mathrm{T}}(G) \leqslant \chi'(G) + k + 1$ where $k$ is an integer such that $k! > n$. A slight modification of the proof gives the following:

**Theorem 6.** *If $G$ is a graph with $n$ vertices and $k$ is an integer with $k!/(2p-1)^k > n$ then $\lambda_p^{\mathrm{T}}(G) \leqslant \chi'(G) + k + 3p - 3$. Hence, as $n \to \infty$, $\lambda_p^{\mathrm{T}}(G) \leqslant \chi'(G) + \mathrm{O}(\log n / \log \log n)$.*

In [6], it is proved that if $\Delta(G)$ is large enough, then $\chi^{\mathrm{T}}(G) \leqslant \Delta(G) + \mathrm{O}(\log^{10} \Delta(G))$ (see also Chapter 9 of [9]). Their proof can easily be modified to show the following result on $(p,1)$-total labelling.

**Theorem 7.** *There exists a $\Delta_0$ such that for $\Delta(G) \geqslant \Delta_0$, $\lambda_p^{\mathrm{T}}(G) \leqslant \Delta(G) + 2\log^{10}(\Delta(G)) + 3p - 2$.*

Molloy and Reed [8] proved that there is a constant $c$ such that the total chromatic number is at most $\Delta + c$ as long as $\Delta$ is sufficiently large, where $c \leqslant 10^{26}$. It is very likely that a similar proof would give an analogous theorem for $(p,1)$-total labelling but with a larger constant.

Following the proof of Theorem 2.1 of [7], one can prove that as $n \to \infty$, the proportion of graphs on vertices $1, 2, \ldots, n$ with $(p,1)$-total number $\lambda_p^{\mathrm{T}} > \Delta + 2p - 1$ is very small. We can state it more precisely as follows:

**Theorem 8.** *Let $q$ and $c$ be constants with $0 < q < 1$ and $0 < c < \min\{\frac{1}{3}, \frac{q}{2}\}$. Then*

$$P\{\lambda_p^{\mathrm{T}}(G_{n,q}) > \Delta + 2p - 1\} = \mathrm{o}(n^{-cn/2}).$$

One approach to prove the $(p,1)$-Total Labelling Conjecture is to obtain a small function $a(p)$ such that a $\Delta + a(p)$ $(p,1)$-total labelling of a graph can be constructed by extending a vertex colouring with a suitable edge colouring.

**Conjecture 9.** Let $p \geqslant 1$. There is an integer $a(p)$, such that for any vertex colouring $c_v$ of a non-complete graph $G$ with colours in $[0, \Delta - 1]$, there is an edge colouring $c_e$ of $G$ with colours in $[0, \Delta + a(p)]$ such that $c_v \cup c_e$ is a $(p,1)$-total labelling of $G$.

Conjecture 9 for $a(p) = 4p - 2$ is implied by the List Colouring Conjecture.

**Definition 10.** Let $G$ be a graph. An *edge list assignment* $L$ is an assignment of a set $L(v)$ of integers to every vertex $v$ of $G$. The graph $G$ is *L-edge colourable* if it admits an application $c$ called *L-edge colouring* from its edge set into the set of integers such that for any edge $e$, $c(e) \in L(e)$ and for any two adjacent edges $e$ and $f$ $c(e) \neq c(f)$. Let $k$ be a non-negative integer. A *k-edge list assignment* is an edge assignment $L$ such that $|L(e)| = k$ for every edge $e$. A graph is *k-edge choosable* if it is $L$-edge colourable for any $k$-edge list assignment $L$. The *list chromatic index* of $G$, denoted $\chi'_l(G)$, is the smallest integer $k$ such that $G$ is $k$-edge choosable.

**List Colouring Conjecture.** The chromatic index is equal to the list chromatic index, that is $\chi' = \chi'_l$.

Since every graph is $(\Delta+1)$-edge colourable (Vizing's theorem), the List Colouring Conjecture implies that it also is $(\Delta+1)$-edge choosable. Let $c_v$ be a vertex colouring of a non-complete graph with colours in $[0, \Delta-1]$. For any edge $e=(x, y)$, there is a set $L(e) \subset [0, \Delta+4p-2]$ of $\Delta+1$ colours such that $L(e) \cap ([c_v(x)-p+1, c_v(x)+p-1] \cup [c_v(y)-p+1, c_v(y)+p-1] = \emptyset$. Then since $G$ is $(\Delta+1)$-choosable, there exists a desired edge colouring.

One can relax the constraints and try to extend the vertex colouring with a fractional edge colouring.

Let $\mathscr{M}$ be the set of matchings of $G$. Given a vertex colouring $c$ with colours in $[1, \Delta-1]$.

We want to minimize the *fractional extend span* $\Delta+p-2+\sum_{M\in\mathscr{M}} w_{\Delta+p-1}(M)$ under the following constraints:

- for $0 \leqslant i \leqslant \Delta+p-2$, $\sum_{M\in\mathscr{M}} w_i(M) \leqslant 1$.
  Each already used colours has a weight at most one on each edge.
- for $e \in E(G)$, $\sum_{e\in M}\sum_{i\in P(e)} w_i(M) \geqslant 1$,
  where $P(e) = [0, \Delta+p-1] \backslash ([c(x)-p+1, c(x)+p-1] \cup [c(y)-p+1, c(y)+p-1])$.
  Each edge must be covered by a weight of one by allowed matching (i.e. with colours at least two apart from the colours of its vertices).

**Theorem 11.** *Let $G$ be a (non-complete) graph. For any vertex colouring $c$ of $G$ with colours in $[0, \Delta-1]$, the fractional extend span is at most $\Delta+3p$.*

**Proof.** Let $M_0, M_1, \ldots, M_\Delta$ be the matching of a $\Delta+1$ edge colouring of $G$. For $0 \leqslant j \leqslant \Delta$, set $w_i(M_j) = 1/(\Delta+1)$ for $0 \leqslant i \leqslant \Delta+p-2$ and $w_{\Delta+p-1}(M_j) = 3p/(\Delta+1)$.

Now we show that the two constraints are satisfied:

For $0 \leqslant i \leqslant \Delta+p-2$, we have

$$\sum_{M\in\mathscr{M}} w_i(M) = \sum_j w_i(M_j) = (\Delta+1)\frac{1}{\Delta+1} = 1.$$

Let $e$ be an edge in one matching $M_{j_e}$.

$$\sum_{e\in\mathscr{M}}\sum_{i\in P(e)} w_i(M) = \sum_{i\in P(e)} w_i(M_{j_e}) = \frac{3p}{\Delta+1} + \frac{1}{\Delta+1}(|P(e)|-1) \geqslant \frac{3p}{\Delta+1} + \frac{\Delta-3p+1}{\Delta+1} \geqslant 1.$$

Then the fractional extend span is at most:

$$\Delta + \sum_{M\in\mathscr{M}} w_{\Delta+1}(M) = \Delta + \sum_j \frac{3p}{\Delta+1} = \Delta+3p. \qquad \square$$

## 3. A $2\Delta - 2\log\Delta$ upper bound

In this section we improve slightly the upper bound $2\Delta+p-1$.

**Theorem 12.** *For any $p \geqslant 1$,*

$$\lambda_p^{\mathrm{T}} \leqslant 2\Delta - 2\log(\Delta+2) + 2\log(16p-8) + p - 1.$$

Obviously this bound is only interesting for "not too large" value of $\Delta$.

*3.1. The tools and ideas*

**Definition 13.** A *cut* $[A, B]$ of a graph $G$ is a set of two induced subgraphs $A$ and $B$ of $G$ such that $(V(A), V(B))$ is a partition of $V(G)$. The bipartite graph $(A, B)$ is the graph with vertex set $V(G)$ and edge set $E(G)\backslash(E(A) \cup E(B))$. The edges of $(A, B)$ are called the *cut edges*. A *maximum cut* $[A, B]$ of $G$ is a cut with the maximum number of cut edges.

**Lemma 14.** *Let $G$ be a graph with maximum degree $2k + 1$. Then a maximum cut $[A, B]$ satisfies $\Delta(A) \leqslant k$ and $\Delta(B) \leqslant k$.*

**Proof.** Consider a maximum cut $[A, B]$. $B$ contains no vertex $b$ of degree greater than $k$ otherwise $[A + b, B - b]$ is a cut with strictly more cut edges. Analogously $A$ has no vertex of degree greater than $k$. $\square$

**Lemma 15.** *Let $G$ be a graph with maximum degree $2k$. Then $G$ has a cut $[A, B]$ such that $\Delta(A) \leqslant k - 1$ and $\Delta(B) \leqslant k$.*

**Proof.** Consider a maximum cut $[A, B]$ which minimizes the number of vertices with degree $k$ in $A$. As in the proof of Lemma 14, $A$ and $B$ contain no vertex of degree greater than $k$. Moreover $A$ has no vertex $a$ of degree $k$, as otherwise $[A - a, B + a]$ is a cut with the same number of cut edges as $[A, B]$ and one vertex less of degree $k$. $\square$

**Lemma 16.** *Let $G$ be a bipartite graph with maximum degree $\Delta$. Then there is an edge colouring $c$ of $G$ in $[1, \Delta]$ such that $c(e) \geqslant i$ only if it is incident to a vertex of degree at least $i$.*

**Proof.** We apply induction on $\Delta$. The result holds trivially for $\Delta = 0$. Consider now a graph with maximum degree $\Delta \geqslant 1$. By König's theorem, it admits an edge colouring $c_1$ in $[1, \Delta]$. Let $M$ be the set of edges coloured $\Delta$ incident to a vertex of degree $\Delta$. Consider $G'$ the graph obtained from $G$ by removing $M$. Since every vertex of degree $\Delta$ is adjacent to an edge of $M$, $\Delta(G') = \Delta - 1$. Then, by induction, $G'$ has an edge colouring $c$ of $G$ in $[1, \Delta - 1]$ such that $c(e) \geqslant i$ only if it is incident to a vertex of degree at least $i$. Extending $c$ into an edge colouring of $G$ in $[1, \Delta]$ by colouring the edges of $M$ with $\Delta$, we obtain the result. $\square$

**Definition 17.** Let $G$ be a graph. A *list-assignment* $L$ is an assignment of a set $L(v)$ of integers to every vertex $v$ of $G$. The graph $G$ is *$L$-colourable* if it admits an application $c$ called *$L$-colouring* from its vertex set into the set of integers such that for any vertex $v$, $c(v) \in L(v)$ and for any edge $(u, v)$, $c(u) \neq c(v)$. Let $v$ be a vertex $G$. A *$(d, v)$-list-assignment* of $G$ is a list-assignment $L$ such that $|L(u)| = d(u)$ if $u \neq v$ and $|L(v)| = d(v) + 1$. We say that $G$ is *$(d, v)$-choosable* if it is $L$-colourable for any $(d, v)$-list-assignment $L$.

**Proposition 18.** *Let $G$ be a connected graph and $v \in V(G)$. Then $G$ is $(d, v)$-choosable.*

**Proof.** There is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $G$ such for $i < n$, the vertex $v_i$ has a neighbour in $\{v_j, i < j \leqslant n\}$. Hence by a greedy algorithm, one can find an $L$-colouring of $G$ for any $(d, v)$-vertex-list-assign ment $L$. $\square$

Using this proposition, we can strengthen Proposition 3.

**Lemma 19.** *Let $G$ be a graph with maximum degree $\Delta \leqslant k$. Then $G$ admits a $(p, 1)$-total labelling in $[0, 2k + p - 1]$ such that a vertex $v$ is assigned a label in $[0, d(v)]$ and an edge is assigned a label in $[k + p - 1, 2k + p - 1]$.*

**Proof.** It suffices to prove it when $G$ is connected.

By Vizing's theorem, there is an edge colouring $c'$ of $G$ with colours in $[k + p - 1, 2k + p - 1]$. Let $v$ be a vertex of $G$ and assume that for every edge $e$ incident to $v$, $c'(e) \geqslant k + p$. Let $L$ be the $(d, v)$-list assignment defined by $L(u) = [0, d(u) - 1]$ if $u \neq v$ and $L(v) = [0, d(v)]$. By Proposition 18, $G$ has an $L$-colouring $c$. The union of $c$ and $c'$ is a $(p, 1)$-total labelling of $G$. As for every edge $e = xy$, if $x \neq v$ then $c(x) \leqslant k - 1 \leqslant c'(e) - p$, and if $x = v$ then $c(v) \leqslant k \leqslant c'(e) - p$. $\square$

Analogously, we have the following lemma:

**Lemma 20.** *Let G be a graph with maximum degree $\Delta \leqslant k$. Then G admits a $(p, 1)$-total labelling in* $[0, 2k + p - 1]$ *such that an edge is assigned a label in* $[0, k]$ *and a vertex v is assigned a label in* $[k + p - 1, k + p - 1 + d(v)]$.

The idea of the proof for Theorem 12 is to consider a suitable maximum cut of $G$ given by Lemma 14 or 15 and to label edges and vertices of $A$ and $B$ with Lemma 19 or by induction hypothesis, and Lemma 20, respectively, and then to label the edges of $(A, B)$ using Lemma 16. Some relabellings are then necessary to obtain the desired $(p, 1)$-total labelling. The following theorem is used.

**Theorem 21** (*Galvin [3]*). *Every bipartite graph G is $\Delta(G)$-edge choosable.*

*3.2. Proof of Theorem 12*

Let $G$ be a graph with maximum degree $\Delta$. A $(p, 1)$-total labelling in $[0, q]$ is a *p-good labelling* if each vertex is assigned a label in $[0, \Delta + p - 1]$.

In order to prove Theorem 12, we shall use induction on $\Delta$ to show that $G$ has a $p$-good labelling in $[0, 2\Delta - 2\log(\Delta + 2) + 2\log(16p - 8) + p - 1]$. Note that Lemma 19 gives the result for small value of $\Delta$. We now give two lemmas allowing us to do an induction step, one for even $\Delta$ and one for odd $\Delta$.

**Lemma 22.** *Let i be an integer and k a positive integer such that* $k \geqslant \max\{i + 2p - 1, 2i + 6p - 5\}$. *If every graph of maximum degree k admits a p-good labelling in* $[0, 2k - i]$, *then every graph G of maximum degree* $\Delta = 2k + 2$ *admits a p-good labelling in* $[0, 2\Delta - i - 2]$.

**Proof.** According to Lemma 15 there is a cut $[A, B]$ of $G$ such that $\Delta(A) \leqslant k$ and $\Delta(B) \leqslant k + 1$. Thus by hypothesis, there is a $p$-good labelling of $A$ in $[0, 2k - i]$. Moreover by Lemma 20, there is a $(p, 1)$-total labelling of $B$ such that vertices are labelled in $[k + p, k + p + d_B(v)]$ and edges in $[0, k + 1]$.

By Lemma 16, label the edges of $(A, B)$ with $[2k - i + 1, 4k - i + 2]$ so that an edge is labelled $4k - i + 3 - l$ only if it is incident to a vertex of degree at least $l$ in $(A, B)$.

The resulting labelling is not yet a $(p, 1)$-total labelling. Indeed for $j \in [0, i + 2p - 1]$, edges $(a, b)$ labelled $2k - i + 1 + j$ when $b$ is labelled in $[2k - i + j - p + 2, 2k - i + j + p]$ violate the constraints. Hence they must be relabelled.

Let us consider the bipartite graph induced by such edges. It has maximum degree at most $i + 2p$. We want to relabel the edges with labels in $[k + 2p - 2, 2k - i]$. According to Theorem 21, it suffices to find a list of $i + 2p$ available labels for each edge. Let $(a, b)$ be an edge labelled $2k - i + 1 + j$ with $b$ labelled in $[2k - i + j - p + 2, 2k - i + j + p]$. Then $d_B(b) \geqslant k - i + j - 2p + 2$. So $b$ has degree at most $k + i - j + 2p$ in $(A, B)$. But by construction $(a, b)$ is incident to a vertex of degree at least $2k + 2 - j$ in $(A, B)$. Since $k \geqslant i + 2p - 1$ then this vertex is $a$ and $d_A(a) \leqslant j$. So at most $j$ labels of $[k + 2p - 2, 2k - i]$ are forbidden because of the edges of $A$ incident to $a$. Moreover at most $2p - j - 2$ labels of $[k + 2p - 2, 2k - i]$ are forbidden because of $b$ (those of $[2k - i + j - 2p + 3, 2k - i]$). Hence at most $2p - 2$ labels of $[k + 2p - 2, 2k - i]$ are forbidden. So because $k \geqslant 2i + 6p - 5$, at least $k - i - 2p + 3 - (2p - 2) \geqslant i + 2p$ labels are available on $(a, b)$.

Since the labels of the vertices are in $[0, 2k + 1 + p]$, we have a $p$-good labelling of $G$ in $[0, 4k - i + 2]$. $\square$

**Lemma 23.** *Let i be an integer and k a positive integer such that* $k \geqslant \max\{i + 4p - 1, 2i + 6p - 3\}$. *If every graph of maximum degree k admits a p-good labelling in* $[0, 2k - i]$ *then every graph G of maximum degree* $\Delta = 2k + 1$ *admits a p-good labelling in* $[0, 2\Delta - i - 2]$.

**Proof.** Let $[A, B]$ be a maximum cut of $G$. Then $\Delta(A) \leqslant k$ and $\Delta(B) \leqslant k$. Thus by hypothesis, there is a $p$-good labelling of $A$ in $[0, 2k - i]$. By Lemma 20, there is a $(p, 1)$-total labelling of $B$ such that vertices are labelled in $[k + p, k + p + d_B(v)]$ and edges in $[1, k]$.

By Lemma 16, label the edges of $(A, B)$ with $[2k - i, 4k - i]$ so that an edge is labelled $4k - i + 1 - l$ only if it is incident to a vertex of degree at least $l$ in $(A, B)$.

There are two types of edges of $(A, B)$ violating a constraint of a $(p, 1)$-total labelling:

(1) edges $(a, b)$ labelled $2k - i + j$ while $b$ is labelled in $[2k - i + j - p + 1, 2k - i + j + p - 1]$ for some $j \in [0, i + 2p - 1]$;
(2) edges $(a, b)$ labelled $2k - i$ with $a$ incident to an edge (of $A$) labelled $2k - i$.

Let us first relabel the edges of type (1) with labels in $[k+2p-1, 2k-i-1]$. Let us consider the bipartite graph induced by them. It has maximum degree at most $i + 2p$. According to Theorem 21, it suffices to find a list of $i + 2p$ available labels for each edge. Let $(a, b)$ be an edge labelled $2k-i+j$ with $b$ labelled in $[2k-i+j-p+1, 2k-i+j+p-1]$. Then $d_B(b) \geqslant k - i + j - 2p + 1$. So $b$ has degree at most $k + i - j + 2p$ in $(A, B)$. But by construction $(a, b)$ is incident to a vertex of degree at least $2k + 1 - j$ in $(A, B)$. Since $k \geqslant i + 2p$, this vertex is $a$ and $d_A(a) \leqslant j$. So at most $j$ labels are forbidden because of the edges of $A$ incident to $a$ and at most $2p - j - 2$ are forbidden because of $b$ (those of $[2k - i + j - 2p + 2, 2k - i - 1]$). Hence at most $2p - 2$ labels of $[k + 2p - 1, 2k - i - 1]$ are forbidden. So since $k \geqslant 2i + 6p - 3$, there are at least $k - i - 2p + 1 - (2p - 2) \geqslant i + 2p$ labels available on $(a, b)$.

Let us now relabel the edges of type (2). Since $a$ is incident to an edge of $A$, it has degree less than $2k + 1$ in $(A, B)$. Hence $b$ has degree $2k + 1$ in $(A, B)$ and thus is isolated in $B$. In particular $b$ was not incident to an edge of type (1). Let $l(a)$ be the label of $a$. There is a label in $[0, k + 2p - 1]\backslash[l(a) - p + 1, l(a) + p - 1]$ that is not assigned to any edge of $A$ incident to $a$. Relabel $(a, b)$ with $l$. Since $l + p \leqslant k + 3p - 1 \leqslant 2k - i - p$, we can relabel $b$ with $k + 3p - 1$.

Since the labels of the vertices are in $[0, 2k + p]$, we have a $p$-good labelling of $G$ in $[0, 4k - i]$. $\square$

Let us now prove by induction on $\Delta$ that $G$ has a $p$-good labelling in $[0, 2\Delta - 2\log(\Delta+2) + 2\log(16p-8) + p - 1]$. Set $c_p = 2\log(16p - 8) + p - 1$. If $\Delta \leqslant 16p - 10$, then we have the result by Lemma 19. Suppose now that $G$ is a graph with maximum degree $\Delta \geqslant 16p - 9$.

Assume that $\Delta$ is even and let $\Delta = 2k + 2$. By induction hypothesis, every graph $H$ with maximum degree $k$ satisfies $\lambda_p^{\mathrm{T}}(H) \leqslant 2k - 2\log(k + 2) + c_p$. Setting $i = 2\log(k + 2) - c_p$, we have $k \geqslant \max\{i + 2p - 2, 2i + 6p - 5\}$. Hence by Lemma 22, $\lambda_p^{\mathrm{T}}(G) \leqslant 2\Delta - 2\log(k + 2) + c_p - 2$. Since $\log(k + 2) + 1 = \log(2k + 4) = \log(\Delta + 2)$, we obtain $\lambda_p^{\mathrm{T}}(G) \leqslant 2\Delta - 2\log(\Delta + 2) + c_p$.

In the same way, we have the result if $\Delta$ is odd. This completes the proof of Theorem 12.

## 4. (2,1)-Total labelling

The upper bound of Theorem 12 is not tight when $\Delta$ is large: it is bigger than the bound given by Theorem 7 which is already bigger than the one expected by the $(p, 1)$-Total Labelling Conjecture. It is not tight for small value of $\Delta$ either. This is due to the fact that the inductive proof of Theorem 12 relies on $p$-good labellings (which are special cases of $(p, 1)$-total labelling). The basis step on the induction is Lemma 19 which is not optimal.

For example, for $p = 2$, Lemma 19 gives a $(2, 1)$-total labelling (actually a 2-good labelling) in $[0, 2\Delta + 1]$. We will show that if $\Delta \geqslant 2$, then $\lambda_2^{\mathrm{T}} \leqslant 2\Delta$.

In particular, if $\Delta = 3$, then $\lambda_2^{\mathrm{T}} \leqslant 6$. Together with Proposition 3 this implies the $(p, 1)$-Total Labelling Conjecture when $\Delta = 3$. Remark also that 6 is tight since $\lambda_2^{\mathrm{T}}(K_4) = 6$ by Proposition 48. However, we think that essentially $K_4$ is the only graph with $\Delta = 3$ and $\lambda_2^{\mathrm{T}} = 6$:

**Conjecture 24.** Let $G$ be a connected graph. If $\Delta(G) \leqslant 3$ and $G \neq K_4$ then $\lambda_2^{\mathrm{T}}(G) \leqslant 5$.

Furthermore, for $\Delta \geqslant 4$, a 2-good labelling in $[0, 2\Delta]$ is shown. So we can improve the bound of Theorem 12, when $p = 2$.

Finally, we will show that if $\Delta$ is odd and at least 7, then $\lambda_2^{\mathrm{T}} \leqslant 2\Delta - 1$.

### *4.1. (2,1)-Total labelling in* $[0, 2\Delta]$

In this subsection, we shall prove the following:

**Theorem 25.** *If* $\Delta \geqslant 2$, *then* $\lambda_2^{\mathrm{T}} \leqslant 2\Delta$.

We divide the proof into three cases: $\Delta = 2$, $\Delta = 3$ and $\Delta \geqslant 4$. In the first two cases (Propositions 26 and 28), we show that the existence of a (2, 1)-total labelling in $[0, 2\Delta]$ which is not 2-good. In the third one (Lemma 30), we show that the existence of a 2-good labelling in $[0, 2\Delta]$. This allows us to improve slightly the upper bound of Theorem 12 when $p = 2$.

**Proposition 26.** *If $\Delta = 2$, then $\lambda_2^{\mathrm{T}}(G) = 4$.*

**Proof.** It suffices to prove it for a connected graph $G$. By Proposition 1, $\lambda_2^{\mathrm{T}}(G) \geqslant 4$. If $G$ is bipartite, then by Corollary 4, $\lambda_2^{\mathrm{T}}(G) \leqslant 4$. Suppose now that $G$ is not bipartite. Then it is an odd cycle $(a_0, a_1, a_2, \ldots, a_{2q}, a_0)$. We have a (2, 1)-total labelling $l$ of $G$ as following: for $1 \leqslant i \leqslant q$, $l(a_{2i-1}a_{2i}) = 4$ and $l(a_{2i}) = 0$, for $1 \leqslant i \leqslant q - 1$, $l(a_{2i+1}) = 1$ and $l(a_{2i}a_{2i+1}) = 3$, and $l(a_0) = 4$, $l(a_1) = 2 = l(a_{2q}a_0) = 2$ and $l(a_0a_1) = 0$. $\square$

**Remark 27.** This proposition shows that the upper bound of the $(p, 1)$-Total Labelling Conjecture is not sharp for some value of $\Delta$ and $p$. Indeed for $\Delta = 2$ and $p = 2$, this conjecture asserts that $\lambda_2^{\mathrm{T}}(G) \leqslant 5$ while $\lambda_2^{\mathrm{T}}(G) = 4$.

**Proposition 28.** *If $\Delta(G) \leqslant 3$, then $\lambda_2^{\mathrm{T}}(G) \leqslant 6$.*

**Proof.** Let $[V_1, V_2]$ be a maximum cut of $G$. By Lemma 14, $\Delta(V_i) \leqslant 1$.

For $i = 1, 2$, let $S_i$ (resp., $T_i$) be the set of isolated vertices (resp., vertices with degree 1) in $G_i$.

Label the edges of $V_1$ (resp., $V_2$) with 3 (resp., 0) and their endvertices with 0 and 1 (resp., 2 and 3). Label the vertices of $S_2$ with 2.

By König's theorem, there is a 3-edge colouring of $(V_1, V_2)$ with colours $a$, $b$ and $c$. For each $a$-coloured edge $(u, v)$ with $u \in G_1$ do the following:

- If $u \in S_1$ and $v \in S_2$, assign 4 to $(u, v)$ and 0 to $u$.
- If $u \in T_1$ and $v \in S_2$, assign 4 to $(u, v)$.
- If $u \in S_1$, $v \in T_2$ and $v$ is labelled 2 then assign 4 to $(u, v)$ and 0 to $u$.

At this stage, the vertices of $S_1$ whose incident $a$-coloured edge has an end in $T_2$ labelled 3 are not yet coloured. We will label them one after another using the following algorithm:

(1) If there is a vertex $y \in T_2$ that is adjacent to two non-labelled vertices $x$ and $z$ (of $S_1$), assign 0 to $x$ and $z$, 3 to $(x, y)$, 4 to $(y, z)$ and relabel $y$ with 6. Go to (1).

(2) If there is a vertex $y \in T_2$ that is adjacent to a non-labelled vertex $x$ and a labelled vertex $z \in S_1$, then $z$ is labelled 0 and there is an integer $l$ in $\{2, 3, 4\}$ that label no edge incident to $z$. Then assign 0 to $x$, $l$ to $(y, z)$, an integer of $\{2, 3, 4\} \setminus \{l\}$ to $(x, y)$ and relabel $y$ with 6. Go to (2).

(3) If there is a vertex $y \in T_2$ that is adjacent to a non-labelled vertex $x$ and a vertex $z \in T_1$. Let $e$ be the edge of $B$ incident to $z$ and distinct from $(y, z)$. If $e$ is not labelled yet then assign 4 to $(y, z)$, 3 to $(x, y)$ and 0 to $x$. Relabel $y$ with 6. Go to (3).
Otherwise $e$ is already labelled with 4. Let $a$ be the label of $z$. Assign 6 to $(y, z)$, 4 to $(x, y)$ and $a$ to $x$. Relabel $y$ with the integer of $\{0, 1\} \setminus \{a\}$. Go to (3).

Let $E'$ be the set of non-labelled edges after this procedure. Clearly, it induces a bipartite graph with maximum degree 2. Moreover the vertices incident to edges of $E'$ are labelled in $[0, 3]$.

By König's theorem, $E'$ can be two coloured with label 5 and 6. It is easy to see that we have a (2, 1)-total labelling of $G$. $\square$

**Alternative proof.** If $G = K_4$, then we have the result by Proposition 48. So we may suppose that $G$ is not complete. Then by Brook's theorem, $\chi(G) = 3$ and $G$ is tripartite. Let $(X, Y, Z)$ be a tripartition of $V(G)$ such that for each $x \in X$, $N(x) \cap Y \neq \emptyset$ and $N(x) \cap Z \neq \emptyset$, and for each $y \in Y$, $N(y) \cap Z \neq \emptyset$.

We will now construct a (2,1)-total labelling of $G$ in three steps:

(1) First assign the label 0, 1, 2, respectively, to the vertices of $X$, $Y$ and $Z$.
(2) Consider $H'$ the graph induced by the edges joining vertices of $Z$ to vertices of $X \cup Y$. It is bipartite and $\Delta(H') \leqslant 3$. Thus, by König's theorem, we may label its edges with the three labels 4, 5 and 6.
(3) Now consider $H$ the graph induced by the edges joining a vertex of $X$ to a vertex of $Y$. By definition of the tripartition, $H$ is bipartite and $\Delta(H) \leqslant 2$. Hence, it is the union of even cycle and paths.
   (a) Let us first label the (even) cycles. Let $C = (a_1, b_1, a_2, b_2, \ldots, a_q, b_q, a_1)$ be a cycle of $H$. For $1 \leqslant i \leqslant q$, assign the label 3 to each edge $a_i b_i$ and label the edge $b_i a_{i+1}$ with the label in {4, 5, 6} which is not used by the two edges joining $b_i$ to a vertex of $Z$ and $a_{i+1}$ to a vertex of $Z$.
   (b) In the same way as in (a), label each odd paths $(a_1, b_1, a_2, b_2, \ldots, a_q, b_q)$.
   (c) Let us now label the even paths one after another. Let $P = (a_1, b_1, a_2, b_2, \ldots, a_q, b_q, a_{q+1})$ be a yet unlabelled even path. For $1 \leqslant i \leqslant q$, assign the label 3 to each edge $a_i b_i$ and for $1 \leqslant i \leqslant q - 1$, assign to the edge $b_i a_{i+1}$ with the label in {4, 5, 6} which is not used by the two edges joining $b_i$ to a vertex of $Z$ and $a_{i+1}$ to a vertex of $Z$. The only edge that remains to be labelled is $e = b_q a_{q+1}$. Therefore, we may need to relabel the vertices $b_q$ and $a_{q+1}$ and the formerly labelled edge $b_q z_0$ where $z_0 \in Z$. Let $z_1$ and $z_2$ be the two neighbours of $a_{q+1}$ in $Z$.
   (i) If there is a label $l \in \{4, 5, 6\}$ which is not used to label $e_0 = b_q z_0$, $e_1 = a_{q+1} z_1$ or $e_2 = a_{q+1} z_2$, then assign $l$ to $b_q a_{q+1}$.
   (ii) If for some $i \in \{0, 1, 2\}$, one of the edges incident to $z_i$ is labelled 0 then one can relabel $e_i$ with a new label in {4, 5, 6} and assign the old one to $e$.
   (iii) If for all $i \in \{0, 1, 2\}$, no edge incident to $z_i$ is labelled 0, then do the following: if $e_0$ is labelled 4, then relabel $e_0$ with 0, $b_q$ with 5 and $a_{q+1}$ with 3 and label $e$ with 1. If not without loss of generality, $e_1$ is labelled 4. Then relabel $e_0$ and $e_1$ with 0, $b_q$ with 5 and $a_{q+1}$ with 3 and label $e$ with 1. $\square$

**Remark 29.** The (2, 1)-total labellings obtained by the two proofs of Theorem 28 are very different.

**Lemma 30.** *If $\Delta(G) \geqslant 4$ then $G$ has a 2-good labelling in* $[0, 2\Delta(G)]$.

**Proof.** Once again, we need to distinguish two cases depending on the parity of $\Delta(G)$.

Suppose first that $\Delta(G)$ is odd, say $\Delta(G) = 2k + 1$. Consider a maximum cut $[A, B]$ of $G$. By Lemma 14, $\Delta(A) \leqslant k$ and $\Delta(B) \leqslant k$.

Thus by Lemmas 19 and 20, one may label $A$ (resp., $B$) in $[0, 2k + 1]$ such that a vertex $v$ in $A$ receives a label in $[0, d_A(v)]$ (resp., $[k + 1, k + 1 + d_B(v)]$) and edges receive labels in $[k + 1, 2k + 1]$ (resp., $[0, k]$).

Now by Lemma 16, label the edges of $(A, B)$ in $[2k + 2, 4k + 2]$ such that an edge is assigned $2k + 2$ only if it is adjacent to a vertex with degree $2k + 1$ in $(A, B)$ and so an isolated vertex in $A$ or $B$.

The label of an edge $(a, b)$ of $(A, B)$ fulfills the constraints of a (2, 1)-total labelling unless it is labelled $2k + 2$ and $b$ is labelled $2k + 1$. But in this case, $a$ is an isolated vertex of $A$ and thus labelled 0. So we may relabel $(a, b)$ with $k + 1$. This is possible since $k \geqslant 2$ so $(2k + 1) - (k + 1) \geqslant 2$.

Since the vertices are labelled in $[0, 2k + 1]$, we have a 2-good labelling. This completes the proof when $\Delta(G)$ is odd.

Suppose now that $\Delta(G)$ is even, say $\Delta(G) = 2k$. Let assume first that $k \geqslant 3$. Consider a cut $[A, B]$ as in Lemma 15. Following Lemma 20, label $A$ such that a vertex $v$ receives a label in $[k+1, k+1+d_A(v)]$ and an edge a label in $[1, k]$. Following Lemma 19, label $B$ such that a vertex $v$ receives a label in $[0, d_B(v)]$ and an edge a label in $[k + 1, 2k + 1]$.

Now by Lemma 16, label the edges of $(A, B)$ in $[2k+1, 4k]$ such that an edge is assigned $2k+1$ only if it is adjacent to a vertex with degree $2k$ in $(A, B)$ and so an isolated vertex in $A$ or $B$.

The label of an edge $(a, b)$ of $(A, B)$ fulfills the constraints of a (2, 1)-total labelling unless $(a, b)$ is labelled $2k + 1$ and (1) $a$ is labelled $2k$ or (2) $b$ is incident to an edge of $B$ labelled $2k + 1$. Thus we need some relabelling.

(1) If $a$ is labelled $2k$, then $a$ is not isolated in $A$. Thus $b$ is isolated in $B$. Then relabel $(a, b)$ with 0 and $b$ with 2.

(2) If $b$ is incident to an edge $(b, b')$ of $B$ which is labelled $2k + 1$, then $b$ is not isolated in $B$. Thus $a$ is isolated in $A$. In particular such an edge is disjoint from any edge of type (1). Let $l(b)$ be the label assigned to $b$. If $l(b) \geqslant 2$ then relabel $(a, b)$ with 0. If $l(b) \leqslant 1$ then relabel $(a, b)$ with 3 and $a$ with 5 if $k = 3$. This is valid since $k \geqslant 3$.

In such a (2, 1)-total labelling a vertex is assigned an integer in $[0, 2k]$, so we have a 2-good labelling.

Suppose now that $k=2$, that is $\Delta(G)=4$. By Lemma 15, $G$ has a cut $[A, B]$ such that $\Delta(A)\leqslant 1$ and $\Delta(B)\leqslant 2$.
Label the vertices of $A$ with $\{0, 1\}$ and its edges with $\{3\}$ such that the isolated vertices of $A$ receive 0.
Label the vertices and edges of $B$ which do not belong to any odd cycle of $B$ as follows:

(i) the isolated vertices of $B$ are labelled 3;
(ii) the vertices (resp., edges) of an even cycle or a path are labelled alternatively 3 and 4 (resp., 0 and 1).

According to Lemma 16, label the edges of $(A, B)$ with [5,8] so that an edge assigned 5 is incident to a vertex of degree 4 in $(A, B)$ which then is an isolated vertex.

Some constraints are violated each time an edge $(a, b)$ of $(A, B)$ is labelled 5 and $a$ is labelled 4. But in that case, $a$ is not isolated in $A$. Thus $b$ is isolated in $B$ and so it is labelled 0. Then relabel $(a, b)$ with 2.

At this stage, it remains to assign labels to vertices and edges of odd cycles of $G$.

Let $C=(b_0, b_1, \dots, b_{2q}, b_0)$ be an odd cycle of $B$. Then two consecutives vertices, say $b_0$ and $b_1$ are either both incident to an edge labelled 5 or both non-incident to an edge labelled 5. Then for $1\leqslant i\leqslant q$, label $b_{2i-1}$ with 3, $b_{2i}$ with 4, $(b_{2i-1}, b_{2i})$ with 1, $(b_{2i}, b_{2i+1})$ with 0 and label $b_0$ with 2. If $b_0$ and $b_1$ are non-incident to an edge labelled 5 then label $(b_0, b_1)$ with 5. Otherwise there is a label $l\in[6, 8]$ such that both $b_0$ and $b_1$ are incident to no edge labelled $l$. Label $(b_0, b_1)$ with $l$. Since the vertices are labelled in $[0, 4]$, we have a 2-good labelling of $G$ in $[0, 8]$. □

Analogously, to the proof of Theorem 12, but using Lemma 30 instead of Lemma 19 as basis of the induction one can get the following:

**Corollary 31.** $\lambda_2^{\mathrm{T}}\leqslant 2\Delta-2\log(\Delta+2)+8$.

*4.2. (2,1)-Total labelling in* $[0, 2\Delta-1]$

Lemmas 30 and 23 immediately yield that if $\Delta$ is odd and at least 19 then $\lambda_2^{\mathrm{T}}\leqslant 2\Delta-1$. We now establish a stronger statement by showing that $\lambda_2^{\mathrm{T}}\leqslant 2\Delta-1$ when $\Delta$ is odd and at least 5.

**Theorem 32.** *If $\Delta$ is odd and at least* 5 *then* $\lambda_2^{\mathrm{T}}\leqslant 2\Delta-1$.

**Proof.** Let $G$ be a graph with maximum degree $\Delta=2k+1\geqslant 5$. We will show a 2-good labelling of $G$ in $[0, 2\Delta-1]$.

Consider a maximum cut $[A, B]$ of $G$. Then $\Delta(A)\leqslant k$ and $\Delta(B)\leqslant k$.

Following Lemma 19, label $A$ such that each vertex $v$ of $A$ is assigned a label in $[0, d_A(v)]$ and each edge $e$ a label in $[k+1, 2k+1]$.

Following Lemma 20, label $B$ such that each vertex $v$ of $B$ is assigned a label in $[k+1, k+1+d_B(v)]$ and each edge $e$ a label in $[0, k]$.

By Lemma 16, label the edges of $(A, B)$ with $[2k+1, 4k+1]$ so that an edge is labelled $4k+2-i$ only if it is incident to a vertex of degree $i$ in $(A, B)$.

This labelling may violate some constraints of a (2, 1)-total labelling in the following cases:

(1) a vertex $b\in B$ labelled $2k$ or $2k+1$ is incident to an edge $(a, b)$ of $(A, B)$ labelled $2k+1$;
(2) a vertex $b\in B$ labelled $2k+1$ is incident to an edge $(a, b)$ of $(A, B)$ labelled $2k+2$;
(3) a vertex $a\in A$ is incident to two edges labelled $2k+1$ one $(a, a')$ in $A$ and one $(a, b)$ in $(A, B)$.

Therefore, we need the following corresponding relabelling:

(1) Since $k\geqslant 2$, then $2k>k+1$ and $b$ is not isolated in $B$. Thus the vertex $a$ is isolated in $A$ and labelled 0. Then relabel $(a, b)$ with $k$.
(2) The vertex $b$ is labelled $2k+1$ and so $d_B(b)=k\geqslant 2$. Hence $b$ has degree less than $2k$ in $(A, B)$ and $a$ has degree at least $2k$ in $(A, B)$. So $a$ has degree at most 1 in $A$ and thus is labelled 0 or 1. One of the two integers $k+1$ and $k+2$ is not used to label the (possible) edge incident to $a$ in $A$. Then relabel $(a, b)$ with $l$. This is valid since $k\geqslant 3$.

(3) Since $a$ is not isolated in $A$, then $b$ is isolated in $B$ and thus labelled $k+1$. If $a$ is labelled 0 or 1 then relabel $(a, b)$ with 3 and $b$ with 5 if $k=3$. Again this is valid since $k \geqslant 3$. If $a$ is labelled in $[2, k+1]$ then relabel $(a, b)$ with 0.

If $k \geqslant 3$, we obtain a 2-good labelling in $[0, 2\Delta - 1]$. However, the last two relabellings are not valid if $k=2$. Hence, to get the result when $\Delta = 5$, we need to be more careful.

Actually, we need a more precise labelling of $A$. Let $C$ be a component of $A$. If $C$ is not an odd cycle, then following Lemma 19, label $C$ such that each vertex $v$ is assigned a label in $[0, d_A(v)]$ and each edge $e$ a label in $[3, 4]$. If $C$ is an odd cycle $(a_1, a_2, \ldots, a_{2q+1}, a_1)$ then for $1 \leqslant i \leqslant q$, label $a_{2i-1}$ with 0, $(a_{2i-1}, a_{2i})$ with 3, $a_{2i-1}$ with 1, and $(a_{2i}, a_{2i+1})$ with 4. Label $a_{2q+1}$ with 2 and $(a_{2q+1}, a_1)$ with 5. Note that in such a labelling a vertex labelled 1 in $A$ is not incident to an edge labelled 5.

Let us now proceed to the relabellings corresponding to the constraints violation (2) and (3).

(2) The vertex $b$ is labelled 5 and so $d_B(b) = 2$. Hence $b$ has degree less than 3 in $(A, B)$ and $a$ has degree at least 4 in $(A, B)$. So $a$ has degree at most 1 in $A$ and thus is labelled 0 or 1. Relabel $(a, b)$ with 3. This may violate a constraint if the edge $(a, a')$ in $A$ incident to $a$ is also labelled 3. If $a'$ is incident to no edge labelled 4 then relabel $(a, a')$ with 4. Otherwise $d_{(A,B)}(a') \leqslant 2$. Thus there is a label $l \in [5, 7]$ that labels no edge incident to $a$ or $a'$ (since $(a, b)$ is now labelled 3). Relabel $(a, a')$ with $l$.

(3) Since $a$ is not isolated in $A$, then $b$ is isolated in $B$ and thus labelled 3. Moreover, the vertex $a$ is labelled either 0 or 2 because no vertex of $A$ labelled 1 is incident to an edge of $A$ labelled 5. If $a$ is labelled 0 then relabel $(a, b)$ with 2 and $b$ with 4. If $a$ is labelled 2 then relabel $(a, b)$ with 0. $\square$

**Remark 33.** Since the (2, 1)-total labelling shown in Theorem 32 is 2-good, we may obtain better upper bounds on $\lambda_2^{\mathrm{T}}$ than the one in Corollary 31 for some values of $\Delta$.

**Remark 34.** The lower bound 5 on $\Delta$ in Theorem 32 is sharp. Indeed the result does not hold when $\Delta = 3$ because $\lambda_2^{\mathrm{T}}(K_4) = 6$ (see Proposition 48). However, Conjecture 24 asserts that the result essentially holds since $K_4$ is the only connected exception.

## 5. The tightness of the bounds

In this section we discuss the tightness of the bounds provided in the previous sections, in particularly, the $(p, 1)$-Total Labelling Conjecture.

If $p \geqslant \Delta + 1$, then the upper bound $2\Delta + p - 1$ of Proposition 3 is attained for the complete graphs:

**Proposition 35.** *If $p \geqslant n$ then $\lambda_p^{\mathrm{T}}(K_n) = 2n + p - 3$.*

**Proof.** By Proposition 3, $\lambda_p^{\mathrm{T}}(K_n) \leqslant 2n + p - 3$.

If $K_n$ admits a $(p, 1)$-total labelling in $[0, 2n+p-4]$. The vertex labels must be in $[0, n-2] \cup [n+p-2, 2n+p-4]$. Indeed, for a vertex with a label in $[n-1, n+p-3]$, at most $(2n+p-4) - (n-1+p) + 1 = n-2$ labels are available for its incident edges and this is a contradiction.

Let $i$ (resp., $2n+p-4-j$) be the largest integer in $[0, n-2]$ (resp., $[n+p-2, 2n+p-4]$) such that a vertex is labelled $i$ (resp., $2n+p-4-j$). Since $n$ different labels are used for the vertices, then $i+j+2 \geqslant n$. Consider the label $l$ of the edge joining the vertices labelled $i$ and $2n+p-4-j$. We have $p+i \leqslant l \leqslant 2n-4-j$. Hence $p \leqslant n-2$ which is a contradiction. $\square$

However, if $p \leqslant \Delta$, the upper bound $2\Delta + p - 1$ of Proposition 3 is not tight for graphs with large maximum degrees. The following result illustrates this clearly.

**Proposition 36.** *Let $G$ be a graph on $n$ vertices, then $\lambda_p^{\mathrm{T}}(G) \leqslant \lambda_p^{\mathrm{T}}(K_n) \leqslant n + 2p - 2$.*

**Proof.** Assign to each vertex $v$ a different integer $l(v)$ from $[0, n-1]$ and assign to an edge $uv$ the integer $l(u) + l(v) + p \bmod n + 2p - 1$. It is clear that two adjacent edges also have different labels since two distinct vertices have different labels. Furthermore, $|l(uv) - l(u) \bmod n + 2p - 1| \geqslant p$. Thus $l$ is a $(p, 1)$-total labelling. $\square$

Proposition 35 shows that the upper bound $\min\{\Delta + 2p - 1, 2\Delta + p - 1\}$ given by the $(p, 1)$-Total Labelling Conjecture is tight when $p \geqslant \Delta + 1$. However as noticed in Remark 27, it is not tight when $p = \Delta = 2$. Moreover, for $p = 2$ and $\Delta = 3$, Conjecture 24 say that the bound is tight but that $K_4$ is the unique connected graph for which it is attained.

Hence one can ask for which value of $p$ and $\Delta$ the upper bound $\min\{\Delta + 2p - 1, 2\Delta + p - 1\}$ is tight and if yes for which graphs.

We will show that when $\Delta = 2$ and $p \geqslant 3$ then the bound of the $(p, 1)$-Total Labelling Conjecture is best possible and attained for odd cycles only. We then show that for $\Delta = p = 3$, the upper bound is not tight.

**Theorem 37.** *Let $G$ be a connected graph with maximum degree 2 and $p \geqslant 3$. If $G$ is an odd cycle then $\lambda_p^{\mathrm{T}}(G) = p + 3$ otherwise $\lambda_p^{\mathrm{T}}(G) = p + 2$.*

**Proof.** If $G$ is not an odd cycle then it is bipartite, so by Corollary 4 and Proposition 1, $\lambda_p^{\mathrm{T}}(G) = p + 2$. Suppose now that $G$ is an odd cycle $(a_0, a_1, a_2, \ldots, a_{2q}, a_0)$. By Proposition 3, $\lambda_2^{\mathrm{T}}(G) \leqslant p + 3$. Suppose for a contradiction that $G$ admits a $(p, 1)$-total labelling in $[0, p + 2]$. Then vertices must be labelled with 0, 1, $p + 1$ or $p + 2$. Since an odd cycle is not 3-colourable, there must be an edge whose endvertices are labelled with one label in $\{0, 1\}$ and one in $\{p + 1, p + 2\}$. Now since $p + 2 < 2p$ this edge may not be labelled. □

By Proposition 3, if $\Delta = 3$ then $\lambda_3^{\mathrm{T}} \leqslant 8$. This shows the $(p, 1)$-Total Labelling Conjecture for $p = \Delta = 3$. But this upper bound is not best possible:

**Theorem 38.** *If $\Delta(G) \leqslant 3$ then $\lambda_3^{\mathrm{T}}(G) \leqslant 7$.*

**Proof.** If $G = K_4$, then we have the result by Proposition 48. So we may suppose that $G$ is not complete. Then by Brook's theorem, $\chi(G) = 3$ and $G$ is tripartite. Let $(X, Y, Z)$ be a tripartition of $V(G)$ such that for each $x \in X$, $N(x) \cap Y \neq \emptyset$ and $N(x) \cap Z \neq \emptyset$, and for each $y \in Y$, $N(y) \cap Z \neq \emptyset$.

Let $H$ be the bipartite graph induced by $X \cup Y$ and $H'$ the graph induced by the edges joining vertices of $Z$ to vertices of $X \cup Y$. The graph $H$ has maximum degree at most 2, so its components are paths and (even) cycles. The graph $H'$ is bipartite and $\Delta(H') \leqslant 3$. Thus, by König's theorem, it is 3-edge colourable. Let $\mathscr{C}$ be the set of edge colourings of $H'$ with colours 5, 6 and 7.

The *ends* of the path $P = (a_1, a_2, \ldots, a_n)$ are the edges $(a_1, a_2]$ and $[a_{n-1}, a_n)$. The different brackets are used to distinguish the endvertices.

Let $c \in \mathscr{C}$ and let $(x, y]$ be an end of an even path of $H$. Let $e_0$ be the edge of $H'$ incident to $y$ and $e_1$ and $e_2$ the edges of $H'$ incident to $x$. We say that $(x, y]$ is *c-good* if $\{c(e_0), c(e_1), c(e_2)\} \neq \{5, 6, 7\}$ or $c(e_0) = 5$. An end that is not $c$-good is said to be *c-bad*. A component of $H$ is *c-bad* if it is an even path (with length at least 2) with two $c$-bad ends.

Let us now consider the edge colouring $c_0 \in \mathscr{C}$ that minimizes the number of bad components in $H$. Let us prove that $c_0$ has no bad paths. Suppose for contradiction that there is a bad path $P_0$. Let $(x_0, y_0]$ be one of its ends and $a$ the colour labelling the edge of $H'$ incident to $y_0$. Since $(x_0, y_0]$ is bad $a \neq 5$ and an edge incident to $x_0$ is labelled 5. Let $Q_0$ be the longest path of $H'$ starting at $x$ with alternating colours 5 and $a$. Let $c_1$ be the edge colouring obtained from $c_0$ by interchanging the colours $a$ and 5 along $Q_0$. Let $z_0$ be the endvertex of $Q_0$ distinct from $x_0$. Since $c_0$ minimizes the number of bad components in $H$, then $c_1$ also minimizes the number of bad components in $H$. Moreover, $P_0$ is $c_1$-good thus $P_1$, the component of $z_0$ in $H$, must be $c_1$-bad and have been $c_0$-good. This implies that $P_1$ is an even path and that $z_0$ belongs to an end of $(x_1, y_1]$ of $P_1$. Furthermore if $z_0 = y_1$, then $c_0(z_0) \neq a$ otherwise $c_1(z_0) = 5$ and $P_1$ is $c_1$-good. In particular, $z_0 \neq y_0$. In addition, $P_0 \neq P_1$ because $(x_0, y_0]$ is $c_1$-good. Set $t_1 = \{x_1, y_1\} \backslash \{z_0\}$.Since $(x_1, y_1]$ is $c_1$-bad, $t_1$ is adjacent to an edge $e_1$ labelled with $a$ or 5. Let $Q_1$ be the longest path of $H'$ starting at $t_1$ with alternating colours 5 and $a$. Let $z_1$ be the endvertex of $Q_1$ distinct from $x_1$, $P_2$ the component of $z_1$ in $H$ and $c_2$ the edge colouring obtained from $c_1$ by interchanging the colours $a$ and 5 along $Q_1$. As before, $c_2$ minimizes the number of bad components and $P_2$ is $c_2$-bad. And $z_1 \neq y_0$. Thus $P_2 \neq P_0$ and because $z_1$ is not in $\{x_1, y_1\}$, $P_2 \neq P_1$. And so on by induction, for any $i \geqslant 0$ one constructs $i$ distinct components of $H$. This is a contradiction since $G$ is finite.

Hence $c_0$ has no bad components.

We will now construct (3,1)-total labelling of $G$ from $c_0$. First assign the label 0, 1, 2 respectively to the vertices of $X$, $Y$ and $Z$. And label the edges of $H'$ according to $c_0$. Let us now label the components of $H$. Let $C$ be such a component.

(a) If $C$ is a cycle $(a_1, b_1, a_2, b_2, \dots, a_q, b_q, a_1)$. For $1 \leqslant i \leqslant q$, assign the label 4 to each edge $a_i b_i$ and label the edge $b_i a_{i+1}$ with the label in $\{5, 6, 7\}$ which is not used by the two edges joining $b_i$ to a vertex of $Z$ and $a_{i+1}$ to a vertex of $Z$.
(b) Proceed analogously if $C$ is an odd path $(a_1, b_1, a_2, b_2, \dots, a_q, b_q)$.
(c) Suppose now that $C$ is the even path $(a_1, b_1, a_2, b_2, \dots, a_q, b_q, a_{q+1})$. By symmetry, we may suppose that $[b_q, a_{q+1})$ is good. For $1 \leqslant i \leqslant q$, assign the label 4 to each edge $a_i b_i$ and for $1 \leqslant i \leqslant q-1$, assign to the edge $b_i a_{i+1}$ with the label in $\{5, 6, 7\}$ which is not used by the two edges joining $b_i$ to a vertex of $Z$ and $a_{i+1}$ to a vertex of $Z$.
Let $z_0$ be the neighbour of $b_q$ in $Z$ and $z_1$ and $z_2$ be the two neighbours of $a_{q+1}$ in $Z$. If there is a label $l \in \{4, 5, 6\}$ which is not used to label $e_0 = b_q z_0$, $e_1 = a_{q+1} z_1$ or $e_2 = a_{q+1} z_2$, then assign $l$ to $b_q a_{q+1}$.
Otherwise since $[b_q, a_{q+1})$ is good, $e_0$ is labelled 5 and $e_1$ and $e_2$ are labelled with 6 and 7. Then relabel $z_0$ with 3, $b_q$ with 7, $a_{q+1}$ with 0 and $e_0$ with 0 and label $b_q a_{q+1}$ with 3.

By construction, this is a (3, 1)-total labelling of $G$. □

## 6. $(p, 1)$-Total labellings of complete graphs

Proposition 36 shows that the $(p, 1)$-Total Labelling Conjecture holds for complete graphs. In this section, we study in more details the $(p, 1)$-total number of complete graphs.

**Proposition 39.** *If $n \geqslant p$, then $\lambda_p^{\mathrm{T}}(K_n) \geqslant n + 2p - 3$.*

**Proof.** Suppose that there is a $(p, 1)$-total labelling with labels in $[0, n+2p-4]$. Let $l$ be a label in $[p-1, n+p-3]$. A vertex cannot be labelled $l$ since there are at most $|[0, n+2p-4]/[l-p+1, l+p-1]| = n-2$ labels allowed for its $n-1$ incident edges. Hence only the $2p-2$ vertices of $[0, p-2] \cup [n+p-2, n+2p-4]$ may be labelled. (In particular, $n \leqslant 2p-2$.) Since $n \geqslant p$, a vertex must get a label in $[0, p-2]$ and another one a label in $[n+p-2, n+2p-4]$. Let $j_1$ be the largest integer in $[0, p-2]$ labelling a vertex $x$ and $n+2p-4-j_2$ be the smallest integer in $[n+p-2, n+2p-4]$ labelling a vertex $y$. The edge $xy$ must be labelled in $[j_1 + p, n + p - 4 - j_2]$. Thus $n + p - 4 - j_2 \geqslant j_1 + p$, so $n \geqslant j_1 + j_2 + 4$. But the labels of all vertices are in $[0, j_1] \cup [n + 2p - 4 - j_2, n + 2p - 4]$. Hence $n \leqslant j_1 + j_2 + 2$ which is a contradiction. □

**Remark 40.** If $n \leqslant p - 1$ then by Theorem 3, $\lambda_p^{\mathrm{T}}(K_n) \leqslant 2\Delta + p - 1 = 2n + p - 3 \leqslant n + 2p - 4$.

Propositions 36 and 39 show that $n + 2p - 3 \leqslant \lambda_p^{\mathrm{T}}(K_n) \leqslant n + 2p - 2$ when $p \leqslant n$. In the rest of the section, we establish the exact value of $\lambda_p^{\mathrm{T}}(K_n)$ between $n + 2p - 3$ and $n + 2p - 2$ for most of the complete graphs.

### *6.1. Odd complete graphs*

**Theorem 41.** *If $n$ is odd then $\lambda_p^{\mathrm{T}}(K_n) \leqslant n + 2p - 3$.*

**Proof.** We will present a labelling using the integers in the interval $[-(n-3)/2 - p, (n-3)/2 + p]$ as the labels.

Consider $K_n$, where $V(K_n) = [-(n-3)/2 - p, -p] \cup \{0\} \cup [p, (n-3)/2 - p]$, which are also the labels of the vertices. Let $F = \{(i, -i), i \in [p, (n-3)/2 + p]\}$. We use 0 to label all edges of $F$.

Before we assign labels to the remaining edges, we partition $K_n - F$ into two isomorphic subgraphs $G_1$ and $G_2$. Furthermore, $G_1 = A_1 \cup B_1$, where $A_1$ is a complete graph on $(n+1)/2$ vertices with the vertex set $[-(n-3)/2 - p, -p] \cup \{0\}$ and $B_1$ is a bipartite graph with bipartitions $[p, (n-3)/2 + p]$ and $[-(n-3)/2 - p, -p]$ and the edge set $\{(i, j) : i + j \in [1, (n-3)/2]\}$.

Clearly, $G_2$ can be considered as the union of $A_2$ and $B_2$ and they are isomorphic to $A_1$ and $B_1$, respectively. We will label the edges of $K_n - F$ in a symmetric manner in the sense that if an edge $e$ in $G_1$ receives the label $i$, then the corresponding edge in $G_2$ receives the label $-i$.

Notice that the edges of $B_1$ are also incident with the vertices in $G_2$. Therefore, the labels used for the edges of $B_1$ will not be used in $G_2$. It is clear that only the vertex 0 is in both $G_1$ and $G_2$ and our labelling strategy for $G_1$ is to assign not only distinct labels to it, but also make sure that if $q$ is a label incident with 0, then $-q$ will not. Then with symmetric manner of the labelling, we will extend the labelling to $G_2$ and obtain a valid one $K_n$.

In $G_1$, we label the edges of $B_1$ first. Notice that the edges of $B_1$ can be partitioned into $(n-3)/2$ matchings, $M_i$, $1 \leqslant i \leqslant (n-3)/2$, where $M_i = \{(j,k) : j+k = (n-1)/2 - i\}$. Hence, $|M_i| = i$.

We assign the labels to the edges of $B_1$ as follows: the edge $(i, j)$ is labelled $i + j$. As we know that $|i|, |j| \geqslant p$, this assignment does not violate the labelling restriction.

For assigning the labels to the edges of $A_1$, we consider two cases.

*Case* 1: $n = 3 \pmod 4$. Let $n = 4k+3$. Consider $K_{2k+2}$, where $V(K_{2k+2}) = \{0, -p, -(p+1), \ldots, -(p+2k)\}$. The edges of $K_{2k+2}$ are labelled as follows:

- If $n < 2p+3$ (or $2k < p$), then we take any 1-factorization of $K_{2k+2}$ and assign the labels $p, p+1, \ldots, 2k+p$ to the $2k+1$ 1-factors (one label for each 1-factor). The labelling is valid because the labels used for $M_i$'s are $1, 2, \ldots, 2k$.
- Otherwise, take a 1-factorization $\{F_1, F_2, \ldots, F_{2k+1}\}$ of $K_{2k+2}$ as described in Lemma 42. We use the labels $p, p+1, \ldots, p+2k$ for the $2k+1$ 1-factors (one label each) as follows. The edges of $F_i$ are labelled $2k-i+1$ for $1 \leqslant i \leqslant 2k-p+1$. For the rest we divide into two subcases.

*Case* 1.1: $p$ is even. For $0 \leqslant i \leqslant (p-2)/2$, $F_{2k-2i}$ is labelled $2k+p-2i$. Assign the rest labels to the unlabelled 1-factors. Clearly, this labelling is not a valid one as it is in conflict with the labels of $M_i$ (the matchings in $B_1$) and may not be extended in a symmetric manner to $G_2$. The vertex $-(p+2k)+j$ is incident to edges labelled $1, 2, \ldots, j$ in $B_1$. Therefore the edges labelled $1, 2, \ldots, j$ incident to it in $A_1$ must be relabelled.

(a) For $1 \leqslant i \leqslant 2k-p+1$, in $F_i$, the edges with both endvertices in $\{0\} \cup (\bigcup_{j=p}^{p+i-1}\{-j\})$ are relabelled $-(2p-i+1)$.

Moreover, to be sure that at most one of the two integers $p$ and $-p$ are used for the edges of $A_1$ incident to 0, some other edges must be relabelled:

(b1) If $k+1 \geqslant p$, for $0 \leqslant i \leqslant (p-2)/2$, in $F_{2k-2i}$, reassign the label $-(2p-2-2i)$ to the edge $(0, -(2k+p-i))$.

(b2) If $k+1 < p$, for $0 \leqslant i \leqslant k-p/2$, in $F_{2k-2i}$, reassign the label $-(2k-2i)$ to the edge $(0, -(2k+p-i))$.

Now all the edges incident to 0 have different labels and if $p$ is one of the labels, then $-p$ is not. Indeed before the relabelling, the labels for the edges incident to 0 are $p, p+1, \ldots, p+2k$. After the relabelling,

- if $k+1 \geqslant p$, then they are $-(2k+p), -(2k+p-2), \ldots, -2p$ (those relabelled with (a)), $-(2p-2), -(2p-4), \ldots, -p$ (those relabelled with (b1)), and $p+1, p+3, \ldots, 2k+3, \ldots, 2k+p-1$ (the non-relabelled ones);
- if $k+1 < p$, the labels are: $-(2k+p), -(2k+p-2), \ldots, -2p$ (those relabelled with (a)), $-2k, -2k+2, \ldots, -p$ (those relabelled with (b2)), $2k+2, 2k+4, \ldots, 2p-2$, and $p+1, p+3, \ldots, 2k+p-1$ (the non-relabelled ones).

Therefore, the labelling we have for $G_1$ is valid. Then we assign labels to $G_2$ in a symmetric manner as described before and we will have a valid labelling we want.

*Case* 1.2: $p$ is odd. For $0 \leqslant i \leqslant (p-3)/2$, $F_{2k-2i}$ is labelled $2k+p-1-2i$. Assign the rest labels to the unlabelled 1-factors. We will again adjust the labels for some of the edges as follows:

(a) For $1 \leqslant i \leqslant 2k-p+1$, in $F_i$, the edges with both endvertices in $\{0\} \cup (\bigcup_{j=p}^{p+i-1}\{-j\})$ are reassigned the label $-(2p-i+1)$.

(b1) If $k+1 \geqslant p$, for $0 \leqslant i \leqslant (p-3)/2$, in $F_{2k-2i}$, reassign the label $-(2p-2-2i)$ to the edge $(0, -(2k+p-i))$.

(b2) If $k+1 < p$, for $0 \leqslant i \leqslant k-(p+1)/2$, in $F_{2k-2i}$, reassign the label $-(2k-2i)$ to the edge $(0, -(2k+p-i))$.

We can verify as before that this labelling is indeed valid.

*Case* 2: $n = 1 \pmod 4$: Let $n = 4k+1$. Consider $K_{2k+1}$, where $V(K_{2k+1}) = \{0, -p, -(p+1), \ldots, -(2k+p-1)\}$. We label the edges of $K_{2k+1}$ as follows.

- If $n \leqslant 2p-1$ (or $2k \leqslant p-1$), then we take any near 1-factorization of $K_{2k+1}$ and assign the labels $p-1, p, \ldots, 2k+p-1$ to the $2k+1$ near 1-factors (one label for each near 1-factor and make sure that the near 1-factor with 0 as the isolated vertex will receive the label $p-1$. Then we are done as this labelling will not be in conflict with the labels assigned to the edges in $M_i$'s or the vertices.

- Otherwise, take a near 1-factorization $\{\mathrm{NF}_1, \mathrm{NF}_2, \ldots, \mathrm{NF}_{2k+1}\}$ of $K_{2k+1}$ as in Lemma 43. First, we use the integers of $[p-1, 2k+p-1]$ to label them: for $2 \leqslant i \leqslant 2k-p+2$. The edges of $\mathrm{NF}_i$ are labelled $2k-i+1$. For the rest, we divide it into two subcases.

The strategy of labelling is the same as in Case 1. Here we will only give the labelling and omit the verification.

*Case* 2.1: $p$ is even. The edges of $\mathrm{NF}_1$ are labelled $2k+p-2$ and for $0 \leqslant i \leqslant (p-4)/2$, assign the label $2k+p-4-2i$ to the edges of $\mathrm{NF}_{2k-2i}$. Then assign the rest labels to the remaining near 1-factors (one label each).

We now adjust the labels for a few edges in order to achieve a valid labelling.

(a) For $3 \leqslant i \leqslant 2k-p+2$, in $\mathrm{NF}_i$, the label of the edges which have both endvertices in the set $\{0\} \cup (\bigcup_{j=p}^{p+i-3}\{-j\})$ is changed to $-(2p+i-3)$. Recall that the original labels for all these edges were: $2k-2, 2k-3, \ldots, p-1$.
(b1) If $k \geqslant p$, for $0 \leqslant i \leqslant (p-4)/2$, in $\mathrm{NF}_{2k-2i}$, reassign the label of $-(2p-2-2i)$ to the edge $(0, -(2k+p-1-i))$.
(b2) If $k < p$, for $0 \leqslant i \leqslant k-p/2-1$, in $\mathrm{NF}_{2k-2i}$, reassign the label $-(2k-2-2i)$ to the edge $(0, -(2k+p-1-i))$.

*Case* 2.2: $p$ is odd. In this case, the edges of $\mathrm{NF}_1$ are labelled $2k+p-1$, and for $0 \leqslant i \leqslant (p-3)/2$, the edges of $\mathrm{NF}_{2k-2i}$ are labelled $2k+p-3-2i$.

(a) For $3 \leqslant i \leqslant 2k-p+2$, in $\mathrm{NF}_i$, the label of the edges which have both endvertices in the set $\{0\} \cup (\bigcup_{j=p}^{p+i-3}\{-j\})$ is changed to $-(2p+i-3)$.
(b1) If $k \geqslant p$, for $0 \leqslant i \leqslant (p-3)/2$, in $\mathrm{NF}_{2k-2i}$, reassign the label $-(2p-2-2i)$ to the edge $(0, -(2k+p-1-i))$.
(b2) If $k < p$, then for $0 \leqslant i \leqslant k-(p+1)/2$, in $\mathrm{NF}_{2k-2i}$, reassign the label $-(2k-2-2i)$ to the edge $((0, -(2k+p-1-i))$. $\square$

**Lemma 42.** *There exists a* 1*-factorization* $\{F_1, F_2, \ldots, F_{2k+1}\}$ *of* $K_{2k+2}$ *with a vertex set* $\{0\} \cup [-p-2k, -p]$ *such that it satisfies the following properties*:

(a) *If i is even*, $F_i$ has $i/2$ *edges covering the vertices* $\{-p, -p-1, \ldots, -p-i+1\}$, if $i \leqslant 2k$,
(b) *If i is odd*, $F_i$ has $(i+1)/2$ *edges covering the vertices* $\{0, -p, \ldots, -p-i+1\}$, if $i \leqslant 2k+1$.

**Proof.** We give an explicit construction of such a 1-factorization. Let $f_i = \{(0, -p-i), (-p-1-i, -p-2k-i), (-p-2-i, -p-2k+1-i), \ldots, (-p-k-i, -p-k-1-i)\}$ for $0 \leqslant i \leqslant 2k$. This is a standard cyclic 1-factorization of $K_{2k+2}$. Now we define $F_i$ as follows.

Let $F_{2i-1} = f_{i-1}$ and $F_{2i} = f_{k+i}$, for $1 \leqslant i \leqslant k$ and $F_{2k+1} = f_k$. We can check that both conditions are satisfied. $\square$

**Lemma 43.** *There exists a near* 1*-factorization* $\{\mathrm{NF}_1, \mathrm{NF}_2, \ldots, \mathrm{NF}_{2k+1}\}$ *of* $K_{2k+1}$ *with a vertex set* $\{0\} \cup [-p-2k+1, -p]$ *such that it satisfies the following properties*:

(a) *If* $i \geqslant 4$ *is even*, $\mathrm{NF}_i$ *has* $i/2-1$ *edges covering the vertex set* $\{-p, -p-1, \ldots, -p-i/2-1\}$ *and the vertex* $-i/2-p$ *is not covered by* $\mathrm{NF}_i$. $\mathrm{NF}_2$ *has* $-p$ *as the isolated vertex.*
(b) *If i is odd*, $\mathrm{NF}_i$ *has* $(i-1)/2$ *edges covering the vertex set* $\{0, -p, \ldots, -p-(i-1)/2\}$ *and the vertex* $-(i+1)/2-p$ *is not covered by* $\mathrm{NF}_i$.

**Proof.** This near 1-factorization can be obtained by deleting the vertex $-p$ from the 1-factorization in Lemma 1 and then relabel the vertex $-p-i$ by $-p-i+1$, for $1 \leqslant p \leqslant 2k$. $\square$

**Corollary 44.** *If n is odd then* $\lambda_p^{\mathrm{T}}(K_n) = \min\{n+2p-3, 2n+p-3\}$.

*6.2. Even complete graphs*

**Theorem 45.** *If n is even and* $n > 6p^2-10p+4$, *then* $\lambda_p^{\mathrm{T}}(K_n) = n+2p-2$.

**Proof.** By Proposition 36, $\lambda_p^{\mathrm{T}}(K_n) \leqslant n+2p-2$.

Let $G$ be a graph on $n$ vertices. Suppose that $G$ admits a $(p,1)$-total labelling with labels in $[0, n+2p-3]$. Then each label $l$ induces a matching $M_l$ over the edges of $G$. Moreover, this label is not adjacent to the vertices with labels in $[l-p+1, l+p-1]$. Let $b(l)$ be the number of labels in $I_l = [l-p+1, l+p-1]$ that are assigned to no vertex. Then $M_l$ contains at most $\lfloor (n-2p+1+b(l))/2 \rfloor = (n-2p)/2 + \lceil b(l)/2 \rceil$ edges and $G$ contains at most $(n+2p-2)(n-2p)/2 + \sum_{i=0}^{n+2p-3} \lceil b(l)/2 \rceil$ edges. Each non-assigned label is contained in $2p-1$ intervals $I_l$. And for $1 \leqslant i \leqslant p-1$ the labels $-i$ and $n+2p-3+i$ are contained in $p-i$ intervals $I_l$. Hence $\sum_{i=0}^{n+2p-3} b(l) \leqslant (2p-2)(2p-1) + 2\sum_{i=1}^{p-1} i = 5p^2 - 7p + 2$. Since $\sum_{i=0}^{n+2p-3} \lceil b(l)/2 \rceil \leqslant \sum_{i=0}^{n+2p-3} b(l)$, if $n > 6p^2 - 10p + 4$, then $G$ has less than $n(n-1)/2$ edges. Thus $G$ is not complete. $\square$

If $p=1$, then $6p^2 - 10p + 4 = 0$. Hence as a corollary, we have the result of Bezhad et al. [1] on total colouring:

**Corollary 46.** $\chi^{\mathrm{T}}(K_n) = \lambda_1^{\mathrm{T}}(K_n) + 1$ *equals $n$ if $n$ is odd, and $n+1$ if $n$ is even.*

**Proposition 47.** *Let $n$ be an even integer greater than 4. If $p \geqslant n-3$, then $\lambda_p^{\mathrm{T}}(K_n) \leqslant n+2p-3$.*

**Proof.** Let us first prove that $\lambda_{n-3}^{\mathrm{T}}(K_n) \leqslant 3n-9$. Label the vertices with $\{0, 1, 2n-7\} \cup [2n-5, 3n-9]$. Since $n > 4$ then $2n-7 > 1$, thus the vertices receive different labels. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-7\} \cup [2n-5, 3n-9]$ with $[0, n-4]$. It is possible since $\chi'(K_{n-2}) = n-3$. For $j \in [2n-5, 3n-9]$, label the edge $(1, j)$ with $j-n+3$ and the edge $(0, j)$ with $j-n+2$. Complete the labelling by assigning $3n-10$ to $(0, 2n-7)$, $3n-9$ to $(1, 2n-7)$, and $3n-8$ to $(0,1)$. One can check that this is a valid $(n-3, 1)$-total labelling of $K_n$. To obtain a $(n-3+i, 1)$-total labelling start from the above labelling and change the label $l$ by $l+i$ if it is in $[n-5, 2n-6]$ and $l+2i$ if it is in $[2n-5, 3n-9]$. $\square$

**Proposition 48.** (i) $\lambda_2^{\mathrm{T}}(K_4) = 6$.
(ii) $\lambda_{3+i}^{\mathrm{T}}(K_4) \leqslant 7+2i$. *In particular,* $\lambda_3^{\mathrm{T}}(K_4) = 7$ *and* $\lambda_4^{\mathrm{T}}(K_4) = 9$.

**Proof.** (i) By Proposition 36, there is a (2,1)-total labelling $l$ of $K_4$ with span 6.

Suppose that there exists an (2,1)-total labelling $l$ of $K_4$ in $[0,5]$. For any vertex $v$, let $A(v)$ be the set of labels of its three incident edges. Now since each vertex must receive a different label, there are two vertices $u$ and $v$ such that $l(u)+1 = l(v)$. Clearly, $|A(u) \cap A(v)| \geqslant 2$ since $l(u)$ and $l(u+1)$ are not contained in both $A(u)$ and $A(v)$. Hence two edges share the same label $l$. Necessarily, there is no vertex labelled $l$, $l-1$ and $l+1$. Since only two labels are not assigned to vertices, either $l=0$ and the four vertices are labelled 2, 3, 4 and 5 or symmetrically $l=5$ and the four vertices are labelled 0, 1, 2 and 3. This implies that only five edges may be labelled which is a contradiction. Indeed in the first case, the label 0 may be assigned to two edges, the labels 1, 2 and 5 to one edge and 3 and 4 to none.

(ii) A $(3+i)$-total labelling in $[0, 7+2i]$ is given by the following adjacency matrix:

| | $0$ | $4+i$ | $6+2i$ | $7+2i$ |
|---|---|---|---|---|
| $0$ | | $7+2i$ | $3+i$ | $4+i$ |
| $4+i$ | $7+2i$ | | $1$ | $0$ |
| $6+2i$ | $3+i$ | $1$ | | $2$ |
| $7+2i$ | $4+i$ | $0$ | $2$ | |

By Proposition 39, $\lambda_3^{\mathrm{T}}(K_4) \geqslant 7$ and $\lambda_4^{\mathrm{T}}(K_4) \geqslant 9$. So $\lambda_3^{\mathrm{T}}(K_4) = 7$ and $\lambda_4^{\mathrm{T}}(K_4) = 9$. $\square$

**Proposition 49.** *Let $n$ be an even integer greater than 5. Then $\lambda_{n-4}^{\mathrm{T}}(K_n) = 3n-11$.*

**Proof.** By Proposition 39, $\lambda_{n-4}^{\mathrm{T}}(K_n) \geqslant 3n-11$.

Let us now show an $(n-4, 1)$-total labelling of $K_n$ in $[0, 3n-11]$. Label the vertices with $\{0, 1, 2n-9\} \cup [2n-7, 3n-11]$. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-9\} \cup [2n-7, 3n-11]$

with $[0, n-4]$ in such a way that the edge $e = (2n-7, 2n-9)$ is labelled $n-4$. It is possible since $\chi'(K_{n-2}) = n-3$. The label of $e$ is not valid. Change it to $3n-11$. For $j \in [2n-7, 3n-11]$, label the edge $(1, j)$ with $j-n+4$ and the edge $(0, j)$ with $j-n+3$. Complete the labelling by assigning $3n-13$ to $(0, 2n-9)$, $3n-12$ to $(1, 2n-9)$, and $3n-11$ to $(0, 1)$. □

**Proposition 50.** Let $n$ be an even integer greater than 7. Then $\lambda^{\mathrm{T}}_{n-5}(K_n) = 3n-13$.

**Proof.** By Proposition 39, $\lambda^{\mathrm{T}}_{n-5}(K_n) \geqslant 3n-13$.

Let us now show an $(n-5, 1)$-total labelling of $K_n$ in $[0, 3n-13]$. Label the vertices with $\{0, 1, 2n-11\} \cup [2n-9, 3n-13]$. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-11\} \cup [2n-9, 3n-11]$ with $[0, n-4]$ in such a way that the edges $e_1 = (2n-11, 2n-9)$ and $e_2 = (2n-11, 2n-8)$ are labelled $n-4$ and $n-5$. The labels of $e_1$ and $e_2$ are not valid. Change them to $3n-14$ and $3n-13$, respectively. For $j \in [2n-7, 3n-13]$, label the edge $(1, j)$ with $j-n+5$ and the edge $(0, j)$ with $j-n+4$. Complete the labelling by assigning $n-3$ to $(1, 2n-8)$, $n-5$ to $(0, 2n-8)$, $n-4$ to $(1, 2n-9)$, $3n-13$ to $(0, 2n-9)$, $3n-15$ to $(1, 2n-11)$, $3n-16$ to $(0, 2n-11)$, and $3n-14$ to $(0, 1)$. By construction, the labels of incident edge and vertex are at distance at least $n-5$. Moreover adjacent edges have different labels if $3n-15 > 2n-8$ that is $n > 7$. □

**Proposition 51.** *Let $n$ be an even integer greater than 7. Then $\lambda^{\mathrm{T}}_{n-6}(K_n) = 3n-15$.*

**Proof.** By Proposition 39, $\lambda^{\mathrm{T}}_{n-6}(K_n) \geqslant 3n-15$.

We give an $(n-6, 1)$-total labelling of $K_n$ in $[0, 3n-15]$ as follows. Label the vertices with $\{0, 1, 2, 3, 2n-11\} \cup [2n-9, 3n-15]$. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-11\} \cup [2n-9, 3n-15]$ with $[0, n-6]$. For $j \in [2n-9, 3n-15]$, label the edge $(3, j)$ with $j-n+6$, the edge $(2, j)$ with $j-n+5$ the edge $(1, j)$ with $j-n+4$ and the edge $(0, j)$ with $j-n+3$. Change the label of $(0, 2n-9)$ into $3n-15$ and label $(0, 2n-11)$ with $n-6$. Complete the labelling by the following labelling of the complete induced by $\{0, 1, 2, 3, 2n-11\}$.

| | 0 | 1 | 2 | 3 | $2n-11$ |
|---|---|---|---|---|---|
| 0 | | $3n-18$ | $3n-17$ | $3n-16$ | $n-6$ |
| 1 | $3n-18$ | | $3n-15$ | $2n-12$ | $3n-17$ |
| 2 | $3n-17$ | $3n-15$ | | $2n-11$ | $3n-16$ |
| 3 | $3n-16$ | $2n-12$ | $2n-11$ | | $3n-15$ |
| $2n-11$ | $n-6$ | $3n-17$ | $3n-16$ | $3n-15$ | |

□

**Problem 52.** What is $\lambda^{\mathrm{T}}_p(K_n)$ when $p+6 \leqslant n \leqslant 6p^2-10p+4$ and $n$ even? $n+2p-3$ or $n+2p-2$?

## 7. Conclusion

In this paper, we have given a number of evidences for the $(p, 1)$-Total Labelling Conjecture to be true. Note that this conjecture implies that $\lambda^{\mathrm{T}}_p(G) \leqslant (1+a)\Delta + (2-a)p - 1$ for any $0 \leqslant a \leqslant 1$. Proposition 3 asserts this for $a = 1$ and it is exactly the $(p, 1)$-Total Labelling Conjecture if $a = 0$. It would be interesting to prove some intermediate results by showing this inequality for some $a < 1$. For example, if $\Delta = 4$, it holds for $a = \frac{1}{2}$ according to Lemma 30 and Proposition 3.

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## References


[1] M. Bezhad, G. Chartand, J.K. Cooper, The colour numbers of complete graphs, J. London Math. Soc. 42 (1967) 226–228.
[2] G.J. Chang, W.-T. Ke, D. Kuo, D.D.-F. Liu, R.K. Yeh, On $L(d, 1)$-labellings of graphs, Discrete Math. 220 (2000) 57–66.
[3] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153–158.
[4] J.R. Griggs, R.K. Yeh, Labelling graphs with a condition at distance two, SIAM J. Discrete Math. 5 (1992) 586–595.
[5] F. Havet, S. Thomassé, Complexity of $(p, 1)$-total labelling, manuscript, 2006.
[6] H. Hind, M. Molloy, B. Reed, Total colouring with $\Delta + \mathrm{poly}(\log \Delta)$ colours, SIAM J. Comput. 28 (1998).
[7] C. McDiarmid, B. Reed, On total colourings of graphs, J. Combin. Theory Ser. B 57 (1993) 122–130.
[8] M. Molloy, B. Reed, A bound on the total chromatic number, Combinatorica 18 (2) (1998) 241–280.
[9] M. Molloy, B. Reed, Graph Colouring and the Probabilistic Method. Algorithms and Combinatorics, vol. 23, Springer, Berlin, 2002.
[10] M.A. Whittlesey, J.P. Georges, D.W. Mauro, On the $\lambda$-number of $Q_n$ and related graphs, SIAM J. Discrete Math. 8 (1995) 449–506.